Chirped Self-Similar Solutions of a Generalized Nonlinear Schrödinger Equation

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An improved homogeneous balance principle and an F-expansion technique are used to construct exact chirped self-similar solutions to the generalized nonlinear Schrödinger equation with distributed dispersion, nonlinearity, and gain coefficients. Such solutions exist under certain conditions and impose constraints on the functions describing dispersion, nonlinearity, and distributed gain function. The results show that the chirp function is related only to the dispersion coefficient, however, it affects all of the system parameters, which influence the form of the wave amplitude. As few characteristic examples and some simple chirped self-similar waves are presented.

Key words: F-Expansion Technique; The Generalized Nonlinear Schrödinger Equation; Chirped Self-Similar Solutions; Propagate Self-Similarly.
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1. Introduction

The (1+1)-dimensional nonlinear Schrödinger equation is one of the most useful generic mathematical models that naturally arises in many fields of physics [1, 2]. Actually, in (1+1) dimensions, it has been proven when a physical system can be expressed by a partial differential equation, then under some suitable approximations, one can always find a nonlinear Schrödinger type equation [3, 4]. This is why the (1+1)-dimensional soliton theory can be successfully used in almost all physical branches.

Recently, studies of exact self-similar solutions to a wide class of nonlinear physical systems have become one of the most exciting and extremely active areas of current investigation, which has been of great value in understanding different nonlinear physical phenomena [5]. Although self-similar solutions have been extensively studied in fields such as hydrodynamics and quantum field theory, their application in optics has not been widespread. Some important results have, however, been obtained, with previous theoretical studies considering self-similar behaviour in radial pattern formation [6], stimulated Raman scattering [7], the evolution of self-written waveguides [8], the formation of Cantor set fractals in soliton systems [9], the nonlinear propagation of pulses with parabolic intensity profiles in optical fibers with normal dispersion [10], and nonlinear compression of chirped solitary waves [11, 12]. In this paper we present the discovery of a broad class of exact self-similar solutions to the nonlinear Schrödinger equation with gain or loss (the generalized NLSE) where all parameters are functions of the distance variable. This class also encloses the set of solitary wave solutions which describes, for example, such physically important applications as the amplification and compression of pulses in optical fiber amplifiers [13]. These linearly chirped solitary wave solutions are applied in the anomalous dispersion regime and may be contrasted with the asymptotic solutions appropriate in the normal dispersion regime [14 – 18]. The importance of the results reported here is twofold: first, the approach leads to a broad class of exact solutions to the nonlinear differential equation in a systematic way. Some of these solutions have been obtained serendipitously in the past, but we emphasize the importance of the use of self-similarity techniques which are broadly applicable for finding solutions to a range of nonlinear partial differential equations, having applications in a variety of other physical situations. These equations are not integrable by the inverse scattering method, and, therefore, from the traditional academic viewpoint they do not have soliton solutions [13]; however, they do have solitary wave solutions which have often been called solitons. The second and more specific significance of these results lies in their potential application to the design of fiber optic amplifiers, optical pulse compressors, and solitary wave based communications links.

2. Exact Self-Similar Solutions of the Generalized Nonlinear Schrödinger Equation with Distributed Coefficients

In this section, we give some exact solutions of the generalized nonlinear Schrödinger equation (NLSE) with distributed coefficients. As is well known, the problem of solution management in the picosecond regime can be described by the generalized NLSE with distributed coefficients [16–20]

$$\mathrm{i}\psi_z - \frac{\beta(z)}{2}\psi_{\tau\tau} + \gamma(z)|\psi|^2\psi = \mathrm{i}\frac{g(z)}{2}\psi, \qquad (1)$$

where $\psi(z,\tau)$ is the complex envelope of the electric field in a co-moving frame, z is the propagation distance, τ is the retarded time, $\beta(z)$ is the group-velocity dispersion parameter, $\gamma(z)$ is the nonlinearity parameter, and $g(z)$ is the distributed gain function. The subscripts z and τ denote the spatial and temporal partial derivatives. It is of note that this model equation has self-similar solutions or quasi-soliton solutions that exhibit a linear chirp. These self-similar pulses or solitary waves are rather stable when propagating along the distance, remaining localized and preserving their sech-type ($\beta(z)\gamma(z) < 0$) or tanh-type ($\beta(z)\gamma(z) > 0$) wave shapes, with only a scaling of wave amplitude and temporal width [1,21]. But here we are concerned with more general types of self-similar solutions such as elliptic double periodic wave solutions rather than only these soliton-like solutions, which are special limited cases of double periodic wave solutions.

To this end, the complex function ψ can be defined in terms of its amplitude and phase

$$\psi(z,\tau) = A(z,\tau)\exp(\mathrm{i}\Phi(z,\tau)), \qquad (2)$$

where $A(z,\tau)$ and $\Phi(z,\tau)$ are real functions. Substituting $\psi(z,\tau)$ into (1), we find the following coupled equations for the phase $\Phi(z,\tau)$ and the amplitude $A(z,\tau)$:

$$A_z - \beta A_\tau \Phi_\tau - \frac{1}{2}\beta A\Phi_{\tau\tau} - \frac{1}{2}gA = 0, \qquad (3)$$

$$A\Phi_z + \frac{1}{2}\beta A_{\tau\tau} - \frac{1}{2}\beta A\Phi_\tau^2 - \gamma A^3 = 0. \qquad (4)$$

Table 1. Jacobi elliptic functions.

Solution	c_0	c_2	c_4	F	$M=0$	$M=1$
1	1	$-(1+M^2)$	M^2	sn	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	cn	cos	sech
3	M^2-1	$2-M^2$	-1	dn	1	sech
4	M^2	$-(1+M^2)$	1	ns	cosec	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	nc	sec	cosh
6	-1	$2-M^2$	M^2-1	nd	1	cosh
7	1	$2-M^2$	$1-M^2$	sc	tan	sinh
8	$1-M^2$	$2-M^2$	1	cs	cot	cosech
9	1	$-(1+M^2)$	M^2	cd	cos	1
10	M^2	$-1+M^2$	1	dc	sec	1

According to the balance principle and F-expansion technique, the solution of (3) and (4) can be expressed in the following form [2]:

$$A(z,\tau) = f_0(z) + f_1(z)F(\theta) + f_2(z)F(\theta)^{-1}, \qquad (5)$$

$$\Phi(z,\tau) = a(z)\tau^2 + b(z)\tau + e(z), \qquad (6)$$

$$\theta = k(z)\tau + l(z), \qquad (7)$$

where $f_0, f_1, f_2, k, l, a, b$, and e are the parameters to be determined. The parameter $a(z)$ is related to the wave front curvature; it is also a measure of the phase chirp imposed on the solitary wave. The function $F(\theta)$ is one of the Jacobi elliptic functions (JEFs), which in general satisfy the following general nonlinear ordinary differential equation:

$$\left(\frac{\mathrm{d}F}{\mathrm{d}\theta}\right)^2 = c_0 + c_2F^2 + c_4F^4, \qquad (8)$$

where c_0, c_2, and c_4 are real constants related to the elliptic modulus of the JEFs (see Table 1). Substituting (5), (6), (7) into (3) and (4) and requiring that t^qF^n ($q = 0,1,2$; $n = 0,1,2,3,4,5,6$), and $\sqrt{c_0 + c_2F^2 + c_4F^4}$ of each term be separately equal to zero, we obtain a system of algebraic or first-order ordinary differential equations for f_p, k, l, a, b, and e:

$$\frac{\mathrm{d}f_p}{\mathrm{d}z} - \left(\beta a + \frac{1}{2}g\right)f_p = 0, \qquad (9)$$

$$f_j\left(\frac{\mathrm{d}k}{\mathrm{d}z} - 2\beta ka\right) = 0, \qquad (10)$$

$$f_j\left(\frac{\mathrm{d}l}{\mathrm{d}z}-\beta kb\right)=0, \tag{11}$$

$$f_j\left(\frac{\mathrm{d}a}{\mathrm{d}z}-2\beta a^2\right)=0, \tag{12}$$

$$f_j\left(\frac{\mathrm{d}b}{\mathrm{d}z}-2\beta ab\right)=0, \tag{13}$$

$$f_0\left(\frac{\mathrm{d}e}{\mathrm{d}z}-\gamma f_0^2-6\gamma f_1f_2-\frac{1}{2}\beta b^2\right)=0, \tag{14}$$

$$f_j\left(\frac{\mathrm{d}e}{\mathrm{d}z}+\frac{1}{2}\beta k^2c_2-3\gamma f_1f_2-3\gamma f_0^2-\frac{1}{2}\beta b^2\right)=0, \tag{15}$$

$$f_1\left(\gamma f_1^2-\beta f_1k^2c_4\right)=0, \tag{16}$$

$$f_2\left(\gamma f_2^2-\beta f_2k^2c_0\right)=0, \tag{17}$$

$$\gamma f_0f_j^2=0, \tag{18}$$

where $p=0,1,2$ and $j=1,2$. By solving self-consistently, one can obtain a set of conditions on the coefficients and parameters, necessary for (1) to have exact periodic wave solutions.

We consider the most generic case, in which f_1 and f_2 are assumed non-zero and β and g are arbitrary. The following set of exact solutions is found:

$$f_0=0,\quad f_1=f_{10}\alpha^{\frac{1}{2}}\exp\left(\int_0^z\frac{1}{2}g\mathrm{d}z\right),f_2=\varepsilon\sqrt{\frac{c_0}{c_4}}f_1; \tag{19}$$

$$k=k_0\alpha,\quad l=l_0+k_0b_0\alpha\int_0^z\beta\mathrm{d}z; \tag{20}$$

$$a=a_0\alpha,\quad b=b_0\alpha; \tag{21}$$

$$e=e_0-\frac{\alpha}{2}\left(k_0^2c_2-6\varepsilon k_0^2\sqrt{c_0c_4}-b_0^2\right)\int_0^z\beta\mathrm{d}z, \tag{22}$$

where $\varepsilon=0,1$ and $\alpha=(1-2a_0\int_0^z\beta\mathrm{d}z)^{-1}$ is the chirp function. It is related to the wave front curvature and present a measure of the phase chirp imposed on the wave since the resultant chirp $\delta\omega(\tau)$ can be expressed by $\delta\omega(\tau)=-\alpha(2a_0\tau+b_0)$. The subscript 0 denotes the value of the given function at $z=0$.

One should note the universal influence of the chirp function α on the solutions. The chirp function is related only to the dispersion coefficient $\beta(z)$; however, it affects all of the parameters. In the case when there is no chirp, $a_0=0$, $\alpha=1$, and the parameters k and b are all constant. In the presence of chirp, they all acquire the prescribed z dependence. The chirp also influences the form of the amplitude A through the dependence of f_1,f_2, and θ on α. It should also be noted that γ is not arbitrary but depends on α,β, and g. The necessary and sufficient condition for existence of such self-similar solutions is given by the following parameter relationship:

$$\gamma=\frac{k_0^2c_4\alpha\beta}{f_{10}^2}\exp\left(-\int_0^z g\mathrm{d}z\right). \tag{23}$$

The constraint condition was firstly suggested by [20], in which α was called the compression/broadening factor. Hence, to obtain exact solutions in a lossy medium (when $g(z)$ is negative), the nonlinearity coefficient γ must grow exponentially. In our choice of independent coefficients, we could have equally well chosen γ and g; then β would have been dependent.

Incorporating the above solutions shown by (19)–(22) back into (2), we obtain the general periodic travelling wave solutions to the generalized NLSE:

$$\psi=f_{10}\alpha^{\frac{1}{2}}\exp\left(\int_0^z\frac{1}{2}g\mathrm{d}z\right)\left[F(\theta)+\varepsilon\sqrt{\frac{c_0}{c_4}}F^{-1}(\theta)\right]\cdot\exp[\mathrm{i}(at^2+bt+e)], \tag{24}$$

where $\theta=k_0\alpha(t+b_0\int_0^z\beta\mathrm{d}z)+l_0$ and $F(\theta)$ is presented in Table 1. Obviously, by selecting different $F(\theta)$ functions, one can derive abundant exact self-similar wave solutions such as periodic wave solutions, double periodic wave solutions, and solitary wave solutions.

3. Wave Self-Similar Propagation – Few Simple Examples

In this part, as few characteristic examples of the solution (24), we present some of the periodic waves and propagating self-similarly soliton solutions, taking the dispersion coefficient β to be of the form $\beta=\beta_0\cos k_bz$ and the gain (loss) coefficient g to be a small constant. This choice leads to alternating regions of positive and negative values of both β and γ, which is required for an eventual stability of localized wave solutions. If the group velocity dispersion parameter β and the distributed gain function g is considered to be $\beta=\cos z$ and $g=0$, then, in Figure 1 we depict two periodic wave solutions made up from the single F functions 1 and 2 from Table 1, with chirp.

Figures 2 and 3 show the limited cases of Figure 1(a) and (b) for $g=0$ and $g=0.04$ giving self-similar solitary waves when the Jacobi module parameter $M\to 1$, respectively.

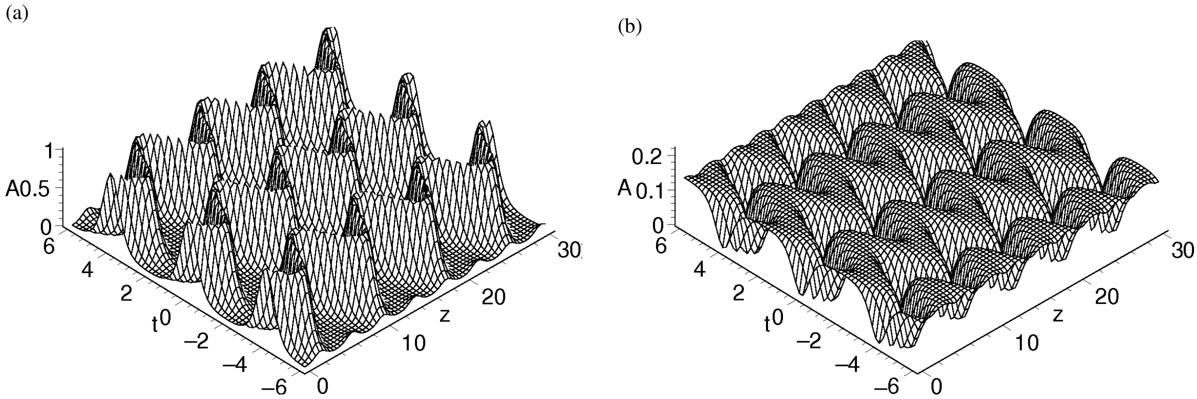

Fig. 1. Periodic travelling wave solutions with chirp, as functions of the propagation distance. (a) show the intensity $|\psi|^2$ of solutions 2 and 1 from Table 1, respectively . Coefficients and parameters: $\beta(z) = \cos z$, $g = 0$, $M = 0.8$, $a_0 = 0.001$, $f_{10} = 1$, $k_0 = 1$, $l_0 = 0$, $\beta_0 = 1$, $k_b = 1$, and $b_0 = 1$.

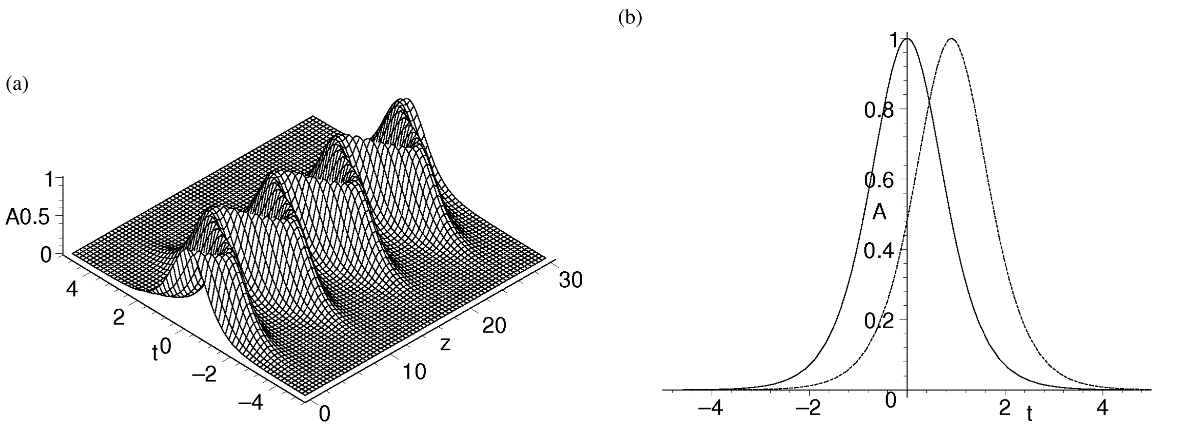

Fig. 2. (a) Propagating solitary wave solutions with chirp, as functions of the propagation distance. Intensity $|\psi|^2$ of solution 2 from Table 1. Coefficients and parameters: $\beta(z) = \cos z$, $g = 0$, $M = 1$, $a_0 = 0.001$, $f_{10} = 1$, $k_0 = 1$, $l_0 = 0$, $\beta_0 = 1$, $k_b = 1$, and $b_0 = 1$; (b) $z = 0$ (solid), $z = 30$ (dash).

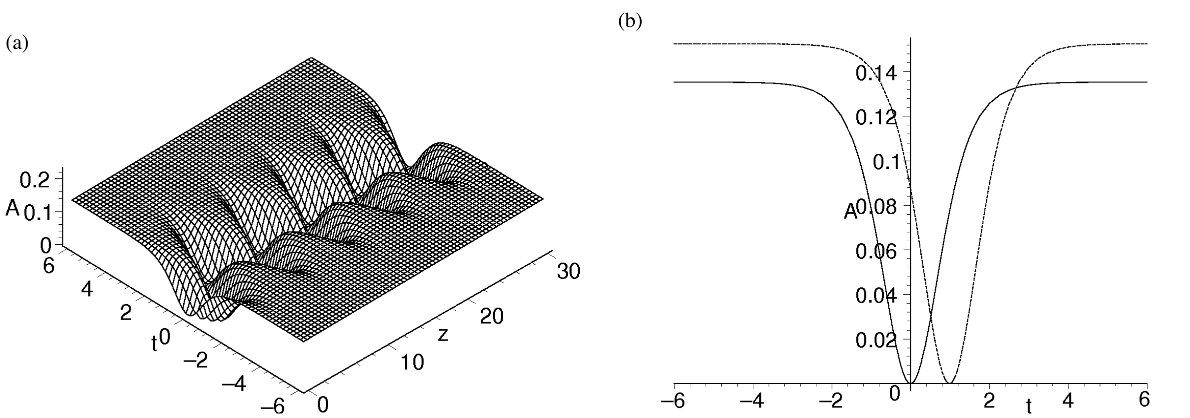

Fig. 3. (a) Travelling solitary wave solutions with chirp, as functions of the propagation distance. Intensity $|\psi|^2$ of solution 1 from Table 1. Coefficients and parameters: $\beta(z) = \cos z$, $g = 0.04$, $M = 1$, $a_0 = 0.001$, $f_{10} = 1$, $k_0 = 1$, $l_0 = 0$, $\beta_0 = 1$, $k_b = 1$, and $b_0 = 1$; (b) $z = 0$ (solid), $z = 30$ (dash).

4. Summary and Conclusion

In summary, an improved homogeneous balance principle and an F-expansion technique are applied to the (1+1)-dimensional generalized nonlinear Schrödinger equation with distributed dispersion, nonlinearity, and gain. Abundant exact self-similar periodic wave solutions are obtained. In some limited cases, different types of soliton solutions are found. A simple and valid procedure is presented for control behaviour of solitons, in which one may select the dispersion and the gain coefficient, or the chirp function and the gain coefficient, to control propagation behaviour of solitons. The present solution method provides a reliable technique that is more transparent and less tedious than the Jacobi elliptic function ansatz, or other expansion and variational methods. The technique is also applicable to other multidimensional nonlinear partial differential equation systems.

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